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SCALE ALTERNATIVES

by
Kjell Doksum

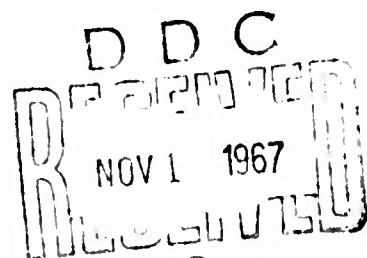
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ABSTRACT

Let X_1, \dots, X_m and Y_1, \dots, Y_n be two independent random samples from populations with continuous IFRA distributions $F(\cdot)$ and $F(\cdot/\Delta)$ respectively, and let s_1, \dots, s_n denote the ranks of the Y 's in the combined sample. For testing $H_0: \Delta \leq 1$ vs. $H_1: \Delta > 1$ it is shown that the error probabilities of each monotone rank test ϕ are bounded by the error probabilities for exponential alternatives, i.e. if $K(t) = 1 - \exp(-t)$, $t \geq 0$, then $P(\text{reject } H_0 | F(\cdot), F(\cdot/\Delta)) \leq P(\text{reject } H_0 | K(\cdot), K(\cdot/\Delta))$ for $\Delta \leq 1$, and $P(\text{accept } H_0 | F(\cdot), F(\cdot/\Delta)) \leq P(\text{accept } H_0 | K(\cdot), K(\cdot/\Delta))$ for $\Delta > 1$. These inequalities are used to derive tests that maximize the minimum power in the class of all rank tests. The results are extended to censored samples, sequential sampling, distributions ordered by skewness, the problem of combining independent test statistics, and the goodness of fit problem.

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1. INTRODUCTION AND SUMMARY

Recent results by Birnbaum, Esary and Marshall (1966), Barlow and Proschan (1967) and others suggest that the exponential models used by Epstein and Sobel (1953) and others for life testing problems should be extended to models in which the lifetimes have increasing failure rate average (IFRA) distributions.

In this paper, IFRA scale models are dealt with. For instance, consider two independent random samples X_1, \dots, X_m and Y_1, \dots, Y_n from populations with distributions $F(\cdot)$ and $F(\cdot/\Delta)$ and respectively, where $F(\cdot)$ is a continuous, unknown, IFRA distribution. The null hypothesis $H_0: \Delta \leq 1$ is to be tested against the alternative $H_1: \Delta > 1$. Let $K(\cdot)$ be the exponential distribution function defined by $K(t) = 1 - \exp(-t)$, $t \geq 0$. It is shown (Theorem 2.1) that for each monotone rank test ϕ and each $\Delta > 1$, the minimum power is attained at the exponential distribution, i.e. $\inf_{F(\cdot)} E[\phi|F(\cdot), F(\cdot/\Delta)] = E[\phi|K(\cdot), K(\cdot/\Delta)]$. Moreover, it is shown that for each $\Delta < 1$, the probability of a type I error is maximized at the exponential distribution, i.e. $\sup_{F(\cdot)} E[\phi|F(\cdot), F(\cdot/\Delta)] = E[\phi|K(\cdot), K(\cdot/\Delta)]$.

Since $F(\cdot)$ is unknown, it is not possible to maximize the power $E[\phi|F(\cdot), F(\cdot/\Delta)]$, $\Delta > 1$. However, it is shown (Theorems 3.1, 3.2, 4.1, 4.2) that the tests that maximize the power for the exponential alternative $K(\cdot/\Delta)$ actually maximize the minimum power $\inf_{F(\cdot)} E[\phi|F(\cdot), F(\cdot/\Delta)]$. Thus these tests are minimax. They have been computed by Lehmann (1953), Savage (1956), and Rao, Savage and Sobel (1960). The results indicate that in the case of uncensored samples, one should use one of the statistics

$$(1.1) \quad L = \sum_{j=1}^n \log [1 - s_j/(N+j)], \text{ or}$$

$$(1.2) \quad S = \sum_{j=1}^n J_o(s_j), \text{ with } J_o(k) = \sum_{j=N+1-k}^n 1/j ,$$

where $N = m + n$ and s_1, \dots, s_n are the ordered ranks of the Y 's in the combined sample of X 's and Y 's. L is minimax for Δ in an interval about two, and S is minimax for Δ in an interval $(1, \Delta)$ to the right of one.

The minimax statistics in the case of censored samples are more complicated (see (4.1) and (4.2)) and one might use one of the approximations suggested by Gastwirth (1965) or Basu (1967) (see (4.3)).

Next, bounds on the error probabilities for sequential rank tests are given in Section 5, and inequalities showing that power is a decreasing function of "skewness to the right" are given in Section 6. Finally, extensions to the problem of combining independent tests statistics are given in Section 7.

Only finite sample size properties are dealt with. Asymptotic results are given in [8].

2. THE POWER OF MONOTONE TESTS

Let X_1, \dots, X_m and Y_1, \dots, Y_n be two independent random samples from populations with distribution functions $F(\cdot)$ and $G(\cdot)$. The problem first considered is that of testing the null hypothesis that the X 's are stochastically larger than or equal to the Y 's against the alternative that the X 's are stochastically smaller than the Y 's; i.e. one tests $H_0: "G(t) \geq F(t) \text{ for all } t,"$ against $H_1: "G(t) \leq F(t) \text{ for all } t \text{ and } G(t) < F(t) \text{ for some } t," -\infty < t < \infty.$

For this problem, it is reasonable to consider tests ϕ that are *monotone* in the sense that

$$(2.1) \quad \begin{aligned} & y_j \leq y_j' \text{ for } j = 1, \dots, n \text{ implies} \\ & \phi(x_1, \dots, x_m; y_1, \dots, y_n) \\ & \leq \phi(x_1, \dots, x_m; y_1', \dots, y_n') \end{aligned}$$

Monotone tests are known (e.g. [12, p. 187] and [4]) to have monotone power, i.e. if $\beta(\phi; F, G) = E[\phi | (F, G)]$ denotes the power of ϕ , then $G_1(\cdot) \leq G_2(\cdot)$ implies $\beta(\phi; G_2, F) \leq \beta(\phi; G_1, F)$. In particular,

Lemma 2.1

If ϕ is monotone, then

$$(2.2) \quad \begin{aligned} & G_1(t) \leq G_2(t) \leq F(t) \text{ for all } t \text{ implies} \\ & \beta(\phi; F, F) \leq \beta(\phi; F, G_2) \leq \beta(\phi; F, G_1) \text{ and} \end{aligned}$$

$$(2.3) \quad \begin{aligned} & F(t) \leq G_1(t) \leq G_2(t) \text{ for all } t \text{ implies} \\ & \beta(\phi; F, G_2) \leq \beta(\phi; F, G_1) \leq \beta(\phi; F, F) \end{aligned}$$

This result shows that for monotone tests, the further the Y 's are away from the X 's stochastically, the smaller are the probabilities of type I and type II errors.

Next the scale model in which $G(t) = F(t/\Delta)$ for all t and some $\Delta > 0$ will be considered. The problem is now to test " $\Delta \leq 1$ " against " $\Delta > 1$." For each test ϕ , let $\beta(\phi; F; \Delta) = E[\phi | F(\cdot), F(\cdot/\Delta)]$ denote the power of ϕ for the scale alternative. $F = F(\cdot)$ is said to be an *IFRA distribution* if $F(0) = 0$ and

$$(2.4) \quad -\log[1 - F(t)]/t \text{ is nondecreasing in } t > 0.$$

Similarly, F is a *DFRA distribution* if $F(0) = 0$ and $-\log[1 - F(t)]/t$ is non-increasing in $t > 0$. Let $s_1 < s_2 < \dots < s_n$ denote the ordered ranks of Y_1, \dots, Y_n in the combined sample and let rank tests be tests depending only on s_1, \dots, s_n .

It will now be shown that for monotone rank tests, the probabilities of type I and type II errors for IFRA distributions are at most what they are for the exponential distribution, i.e. if $K(t) = 1 - \exp(-t)$, $t \geq 0$, then

Theorem 2.1

If ϕ is a monotone rank test and F is a continuous IFRA distribution, then

$$(2.5) \quad \text{for each } \Delta \geq 1, \beta(\phi; K; \Delta) \leq \beta(\phi; F; \Delta), \text{ and}$$

$$(2.6) \quad \text{for each } 0 < \Delta \leq 1, \beta(\phi; F; \Delta) \leq \beta(\phi; K; \Delta).$$

Proof:

Since $B(t) = -\log[1 - F(t)]$ is an increasing function and since the ranks s_1, \dots, s_n are invariant under increasing transformations, then

$$\begin{aligned}
 & \phi(X_1, \dots, X_m; Y_1, \dots, Y_n) \\
 (2.7) \quad & = \phi(X_1', \dots, X_m'; Y_1', \dots, Y_n') \\
 & \text{where } X_i' = B(X_i) \text{ and } Y_j' = B(Y_j).
 \end{aligned}$$

Next note that if one defines $Y_j'' = -\Delta \log \left[1 - F\left(\frac{Y_j'}{\Delta}\right) \right]$, then the definition of the IFRA property implies that for each $\Delta \geq 1$,

$$(2.8) \quad \frac{Y_j''}{Y_j'} = -\frac{1}{(Y_j'/\Delta)} \log[1 - F(Y_j'/\Delta)] \leq -\frac{1}{Y_j'} \log[1 - F(Y_j')] = \frac{Y_j'}{Y_j'}.$$

That is, $Y_j'' \leq Y_j'$, and since ϕ is monotone, then

$$\begin{aligned}
 & \phi(X_1', \dots, X_m'; Y_1'', \dots, Y_n'') \\
 (2.9) \quad & \leq \phi(X_1', \dots, X_m'; Y_1', \dots, Y_n')
 \end{aligned}$$

Upon using (2.7), taking expectations in (2.9), and noting that X_1' has the distribution $K(\cdot)$ while Y_j'' has the distribution $K(\cdot/\Delta)$, one obtains (2.5). The inequalities are reversed for $0 < \Delta \leq 1$, and (2.6) follows.

For DFRA distributions, the opposite result holds, i.e.

Lemma 2.2

If ϕ is a monotone rank test and F is a continuous DFRA distribution, then

$$(2.10) \quad \text{for each } \Delta \geq 1, \quad \beta(\phi; F; \Delta) \leq \beta(\phi; K; \Delta), \text{ and}$$

$$(2.11) \quad \text{for each } 0 < \Delta \leq 1, \quad \beta(\phi; K; \Delta) \leq \beta(\phi; F; \Delta)$$

The proof is the same as that of Theorem 2.1 with the inequalities reversed.

Remark

Lemma 2.1 shows that if ϕ is monotone, then $\beta(\phi; F; \Delta)$ is an increasing function of $\Delta > 0$. Thus Theorem 2.1 implies that the power of monotone rank tests has the bound $\beta(\phi; F; \Delta) \geq \beta(\phi; K; \Delta_1)$ for the IFRA scale alternative with $\Delta \geq \Delta_1 \geq 1$.

3. MINIMAX TESTS IN THE TWO-SAMPLE CASE

Although it is a desirable property for a test to have monotone power for the stochastic ordering alternative H_1 of Section 2, this alternative cannot be used to derive optimal tests. One would like to have tests that are most likely to reject for alternatives that indicate a definite distinction between the distributions of X and Y . When X and Y are time measurements, then one such model is the scale model of Section 2 in which Y has the same distribution as ΔX for some $\Delta > 0$. Then $[E(Y) - E(X)]/E(X) = \Delta - 1$, and $(\Delta - 1)$ measures the relative difference of the mean times.

Note that if $N = m + n$ and if s'_j is defined to be the number of Y 's greater than or equal to the $(N+1-j)$ th order statistic in the combined sample, then (s'_1, \dots, s'_N) is equivalent to the ordered ranks (s_1, \dots, s_n) . Next it will be shown that in the class of level α rank tests, the Lehmann level α test ϕ_Δ which rejects for small values of

$$(3.1) \quad \sum_{j=1}^N \log \left[j - \frac{(\Delta-1)}{\Delta} s'_j \right]$$

maximizes the minimum power $\inf_F \beta(\phi; F; \Delta)$ for the IFRA scale model whenever $\Delta > 1$. Here, a rank test ϕ is said to be of level α if $\beta(\phi; F; F) = E(\phi | (F, F)) = \alpha$ for all continuous distribution functions F . As in Section 2, $\beta(\phi; F; \Delta) = E[\phi | F(\cdot), F(\cdot/\Delta)]$ denotes the power of ϕ for the scale alternative.

Theorem 3.1

The Lehmann test ϕ_Δ is minimax for the IFRA scale alternative in the sense that for each scale parameter $\Delta > 1$ and for F ranging over the class of continuous IFRA distributions,

$$(3.2) \quad \inf_F \beta(\phi; F; \Delta) \leq \inf_F \beta(\phi_\Delta; F; \Delta)$$

for each level α rank test ϕ . Moreover, ϕ_Δ is the unique minimax rank test in the sense that any other level α rank test satisfying (3.2) coincides with ϕ_Δ a.e.

Proof:

Using the Neyman-Pearson Lemma, Lehmann (1953), Savage (1956), and Rao, Savage and Sobel (1960, Corollary 3.4) have essentially shown that ϕ_Δ maximizes the power for the exponential scale model, i.e. for each $\Delta > 1$,

$$(3.3) \quad \beta(\phi; K; \Delta) \leq \beta(\phi_\Delta; K; \Delta)$$

for all level α rank tests ϕ . On the other hand, the definition (3.1) shows that ϕ_Δ is monotone, thus Theorem (2.1) implies that ϕ_Δ attains its minimum power for the exponential scale model, i.e.

$$(3.4) \quad \inf_F \beta(\phi_\Delta; F; \Delta) = \beta(\phi_\Delta; K; \Delta)$$

Since K is an IFRA distribution, then

$$(3.5) \quad \inf_F \beta(\phi; F; \Delta) \leq \beta(\phi; K; \Delta)$$

and (3.2) follows. Uniqueness follows also since if ψ is a level α rank test satisfying (3.2), then the above shows that it must satisfy

$$(3.6) \quad \beta(\phi; K; \Delta) \leq \beta(\psi; K; \Delta)$$

for all level α rank tests ϕ . Since ϕ_Δ also satisfies this (see (3.3)), then the uniqueness part of the Neyman-Pearson Lemma implies that $\psi = \phi_\Delta$ a.e.

In order to be able to use the minimax test ϕ_Δ , one must choose a value of Δ . Savage (1956) suggests using the level α test ϕ_s that maximizes the power for the exponential model for Δ in a neighborhood $(1, \delta)$ to the right of one, i.e.

when the relative difference of the means of X and Y is close to zero (and positive). This test reject for large values of

$$(3.7) \quad \sum_{j=1}^n J_0(s_j), \text{ where } J_0(k) = \sum_{j=N+1-k}^N 1/j.$$

Note that $J_0(k)$ can be approximated by $-\log[1-k/(N+1)]$. It can now be shown that the Savage test ϕ_s is minimax for Δ in an interval of the form $(1, \delta)$.

Theorem 3.2

The Savage test ϕ_s is locally minimax for the IFRA scale alternative in the sense that there exists $\delta > 1$ such that for F ranging over the class of continuous IFRA distributions,

$$(3.8) \quad \inf_F \beta(\phi; F; \Delta) \leq \inf_F \beta(\phi_s; F; \Delta)$$

for each level α rank test ϕ and for all Δ in the interval $(1, \delta)$. Moreover, ϕ_s is the unique locally minimax rank test in the sense that any other level α rank test satisfying (3.8) coincides with ϕ_s a.e.

Proof:

Savage (1956) has essentially shown that there exists $\delta > 1$ such that

$$(3.9) \quad \beta(\phi; K; \Delta) \leq \beta(\phi_s; K; \Delta)$$

for all level α tests ϕ and all Δ in $(1, \delta)$. Since ϕ_s is monotone, the remainder of the proof is as the proof of Theorem 1.

Remarks

(1) The Savage test maximizes the minimum power when the relative difference of the means of X and Y ($\Delta-1$) is close to zero. In most situations, it would be better to use the test that is optimal when $(\Delta-1)$ is in a neighborhood of some

fixed positive number λ (say). Thus one would use the Lehmann test ϕ_Δ defined by (3.1) with $\Delta = \lambda + 1$. For instance, if the relative difference of the means of X and Y is unity, then Lehmann (1953) has shown that $\phi_\Delta = \phi_2$ is equivalent to the test that rejects for small values of

$$(3.10) \quad \sum_{j=1}^n (N+j-s_j) \quad \text{or} \quad \sum_{j=1}^n \log \left(1 - \frac{s_j}{N+j} \right).$$

(ii) The uniqueness parts of Theorem 3.1 and 3.2 can be extended as follows. A test ϕ is said to be *distribution-free* (DF) if $\beta(\phi; F, F) = E(\phi | (F, F))$ is independent of F for F continuous. Thus rank tests are DF. For Δ close to one, tests that are not DF have minimum power less than ϕ_Δ and ϕ_s , and they cannot be minimax. To see this, note that if ϕ is of level α , a.e. continuous, and not DF, then there exists a continuous distribution F_0 such that $E[\phi | F_0, F_0] < \alpha = \sup_F E[\phi | (F, F)]$. Now for Δ close to one $\beta(\phi; F_0, \Delta) < \alpha$ and ϕ is worse than ϕ_Δ and ϕ_s .

(iii) Lemma 2.1 shows that if ϕ is a monotone test, then $\beta(\phi; F; \Delta)$ is an increasing function of the scale parameter $\Delta > 0$. This implies that for fixed $\Delta_1 > 1$, ϕ_{Δ_1} is doubly minimax for the scale alternative in the sense that for testing $H_0: \Delta \leq 1$ against $H_1: \Delta \geq \Delta_1$, it maximizes $\inf_F [\inf_{\Delta \geq \Delta_1} \beta(\phi; F, \Delta)] = \inf_{\Delta \geq \Delta_1} [\inf_F \beta(\phi; F, \Delta)]$. Here, F ranges over the class of continuous IFRA distributions and only level α rank tests are considered. See the remark at the end of Section 2. Note that $(1, \Delta_1)$ is an indifference region.

4. MINIMAX TESTS BASED ON CENSORED SAMPLES

The censored samples considered here arise typically as follows: m objects of one type and n objects of a second type are put on trial at the same time and one waits until a total of $N^* < m + n = N$ of the objects have failed, where N^* is a fixed number. Moreover the experiment is conducted so that if one waited until all $N = m + n$ objects failed, then the times to failure X_1, \dots, X_m and Y_1, \dots, Y_n would be two independent random samples.

Thus the situation is as in Sections 2 and 3 except that only the first N^* smallest order statistics in the combined sample are observed. Let m^* and n^* denote the total number of X 's and Y 's observed respectively. Since the unobserved X 's and Y 's are all larger than the observed ones, it is possible to compute the ordered ranks $s_1 < \dots < s_{n^*}$ of Y_1, \dots, Y_{n^*} among the uncensored combined sample $X_1, \dots, X_m; Y_1, \dots, Y_{n^*}$. Rao, Savage and Sobel (1960, Corollary 3.4) have computed the most powerful test ϕ_Δ^* based on s_1, \dots, s_{n^*} for exponential alternatives. This level α test rejects H_0 for small values of

$$(4.1) \quad \sum_{j=1}^{N^*} \log \left[A(\Delta) + j - \frac{(\Delta-1)}{\Delta} s_j' \right] + n^* \log \Delta$$

with $A(\Delta) = (m - m^*) + (n - n^*)/\Delta$

where s_j' is as in Section 3. For tests depending only on s_1, \dots, s_{n^*} one obtains

Theorem 4.1

The test ϕ_Δ^ is uniquely minimax for the IFRA scale alternative in the sense of Theorem 3.1.*

Proof:

It is enough to show that ϕ_Δ^* is monotone and then apply the proof of Theorem 3.1.

However, it is clear from the definition (4.1) that ϕ_s^* is monotone since $A(\Delta)$ $-s_j^*$ and n^* are decreasing functions of the y 's.

The locally most powerful test ϕ_s^* (Rao, Savage and Sobel (1966, Corollary 3.4)) for the exponential alternative $K(\cdot/\Delta)$ rejects for large values of

$$(4.2) \quad \sum_{j=1}^{n^*} J_o^*(s_j) + (n-n^*)J_o^*(N^*) - n^*$$

where $J_o^*(k) = \sum_{j=N^*-k+1}^{N^*} 1/[j + (m-m^*) + (n-n^*)]$

Theorem 4.2

The test ϕ_s^* is uniquely locally minimax for the IFRA scale alternative in the sense of Theorem 3.2.

Proof:

Again, it is enough to show that ϕ_s^* is monotone. Monotonicity follows at once from (4.2) and the inequality $J_o^*(s_j) \leq J_o^*(N^*)$, $j = 1, \dots, n^*$.

Remark

The tests ϕ_Δ^* and ϕ_s^* are complicated and one may use the approximations suggested by Gastwirth (1965) and Basu (1967). The latter's paper contains tables of rejection points and power for the test based on the statistic

$$(4.3) \quad \sum_{j=1}^{n^*} J_o(s_j) + \frac{n-n^*}{N-N^*} \sum_{j=N^*+1}^N J_o(j) - \frac{1}{2} (m+n)$$

$$= \sum_{j=1}^{n^*} J_o(s_j) + (n-n^*) J_o(N^*) - n^* + \frac{1}{2} (m-n)$$

where as in Section 3, $J_o(k) = \sum_{j=N-k+1}^N 1/j$.

5. SEQUENTIAL TESTS

Rao, Savage and Sobel (1960), Sobel (1966), Basu (1967) and others have pointed out that if one is measuring time variables, then all rank tests have interpretations as truncated (curtailed) sequential tests. Consider for instance the uncensored samples of Section 3. If the significance level α is .10, if $m = n = 5$ and if the first four values observed are X 's, then one can stop the experiment since H_0 will be rejected no matter what the remainder of the observations turns out to be. Thus only four observations are needed. The maximum number of observations needed until a decision is reached (the truncation point) is ten. Tables for carrying out some truncated sequential rank tests and tables containing expected sample sizes and sample times are given by Sobel (1966) and Basu (1967).

The situation considered here is the one in which the observations at the n th stage form two independent random samples $X_1, \dots, X_n; Y_1, \dots, Y_n$. A sequential test ϕ is monotone if

$$\begin{aligned}
 & y_j \leq y'_j \text{ for } j = 1, 2, \dots \text{ implies} \\
 (5.1) \quad & \phi(x_1, x_2, \dots; y_1, y_2, \dots) \\
 & \leq \phi(x_1, x_2, \dots; y'_1, y'_2, \dots)
 \end{aligned}$$

From the arguments of the proof of Theorem 2.1, it follows that for the IFRA scale model, the error probabilities of each monotone sequential rank test is bounded by the error probability for the exponential alternative $K(t/\Delta) = 1 - \exp(-t/\Delta)$.

Here rank tests are tests which at the n th stage depend on the ordered ranks

s_1, \dots, s_n of the Y 's in the combined sample $X_1, \dots, X_n; Y_1, \dots, Y_n$. As before, s'_j is the number of Y 's greater than or equal to the $(N+1-j)$ th order statistic in the combined sample. Thus (s'_1, \dots, s'_{2n}) is equivalent to (s_1, \dots, s_n) . $\beta_s(\phi; F; \Delta)$ denotes the power of ϕ when the X 's and Y 's have distributions $F(\cdot)$ and $F(\cdot/\Delta)$ respectively.

Theorem 5.1

For each monotone sequential rank test ϕ and each continuous IFRA distribution F

$$(5.2) \quad \Delta \leq 1 \text{ implies } \beta_s(\phi; F; \Delta) \leq \beta_s(\phi; K; \Delta) \text{ and,}$$

$$(5.3) \quad \Delta \geq 1 \text{ implies } \beta_s(\phi; F; \Delta) \geq \beta_s(\phi; K; \Delta)$$

Proof:

Same as the proof of Theorem 2.1.

The sequential probability ratio test $\hat{\phi} = \hat{\phi}(a, b)$ based on ranks for the exponential alternative $K(\cdot/\Delta)$ has been studied by Parent (1965), Savage and Sethuraman (1966) and others. For testing $\Delta = 1$ against $\Delta = \Delta_1 > 1$, it can be written

Take two more observations (X_{n+1}, Y_{n+1}) if

$$\log(a) \leq L_n \leq \log(b),$$

$$(5.4) \quad \text{accept } H_0 \text{ if } L_n < \log(a),$$

$$\text{reject } H_0 \text{ if } L_n > \log(b),$$

$$n = 1, 2, \dots$$

where $0 < a < 1 < b$ are constants independent of n and

$$(5.5) \quad L_n = \log[(2n)!] - n \log \Delta_1 - \sum_{j=1}^N \log \left[j - \frac{(\Delta_1 - 1)}{\Delta_1} s_j' \right]$$

It is clear that this test is monotone, thus the bounds on the error probabilities given in Theorem 5.1 apply. Let α_0 and α_1 be desired error probabilities for $\Delta = 1$ and $\Delta = \Delta_1$ respectively, and let a and b be such that the test $\hat{\phi}(a, b)$ achieve these bounds for the exponential distribution, i.e.

$$\beta_s(\hat{\phi}; K; 1) = \alpha_0 \text{ and } \beta_s(\hat{\phi}; K; \Delta_1) = \alpha_1. \text{ Then Theorem 5.1 and the sequential version}$$

of Lemma 2.1 implies that α_0 and α_1 are bounds on the error probabilities for all IFRA distributions for $\Delta \leq 1$ and $\Delta \geq \Delta_1$ respectively; i.e.

$$\begin{aligned} \beta_s(\hat{\phi}; F; \Delta) &\leq \alpha_0 \quad \text{for } \Delta \leq 1, \text{ and} \\ \beta_s(\hat{\phi}; F; \Delta) &\geq 1 - \alpha_1 \quad \text{for } \Delta \geq \Delta_1 \end{aligned} \quad (5.6)$$

for all continuous IFRA distributions F .

Note that the test $\hat{\phi}$ is easy to carry out as a and b can be closely approximated as follows

$$(5.7) \quad a \doteq \frac{\alpha_1}{1 - \alpha_0} \quad \text{and} \quad b \doteq \frac{1 - \alpha_1}{\alpha_0}$$

Since $\hat{\phi}$ is not known to be optimal for the exponential alternative $K(\cdot/\Delta)$, it is not possible to prove any minimax optimality properties of this test.

6. GENERALIZATIONS, SKEWNESS

The results of the previous sections hold if the class of continuous IFRA distributions is replaced by any class of continuous distributions that is contained in this class and contains the exponential distribution; e.g. the class of continuous IFR (increasing failure rate) distributions.

A distribution $F(\cdot)$ is said to be IFR if $K^{-1}[F(\cdot)]$ is convex, where $K(t) = 1 - \exp(-t)$, $t \geq 0$. Extending this, van Zwet (1964) calls a distribution $F(\cdot)$ *less skew* (to the right) than $H(\cdot)$ if $H^{-1}[F(\cdot)]$ is convex, and he writes $F \leq_c H$. Thus a distribution is IFR if it is less skew than the exponential distribution $K(\cdot)$. Carrying these ideas further, we define a distribution $F(\cdot)$ to be *weakly less skew* (to the right) than H if $H^{-1}[F(\cdot)]$ is starshaped ($H^{-1}[F(t)]/t$ is non-decreasing in $t > 0$), and write $F \leq_{\star} H$. Thus a distribution is IFRA if it is weakly less skew than the exponential distribution $K(\cdot)$. One can now define F to be *more IFRA* than H if $F \leq_{\star} H$. It is assumed throughout this section that all the distributions satisfy $F(0) = 0$ and have inverses on the interval $(0, \infty)$. From the properties of starshaped and convex functions it follows that

Lemma 6.1

- (i) If F is weakly less skew than H , and H is weakly less skew than L , then F is weakly less skew than L .
- (ii) If F is less skew than H , then F is weakly less skew than H .

Next one shows that for scale alternatives, the error probabilities of each monotone rank test ϕ are decreasing functions of weak skewness (to the right), or equivalently, increasing functions of the IFRA property.

Theorem 6.1

If ϕ is a monotone two-sample rank test and if F and H are continuous

distributions such that F is weakly less skew than H , then

$$(i) \quad \text{for each } \Delta \leq 1, \quad \beta(\phi; F; \Delta) \leq \beta(\phi; H; \Delta)$$

$$(ii) \quad \text{for each } \Delta \geq 1, \quad \beta(\phi; H; \Delta) \leq \beta(\phi; F; \Delta)$$

Proof:

The proof is the same as for Theorem 2.1 with $H^{-1}[F(\cdot)]$ replacing $-\log[1-F(\cdot)]$.

One can now proceed as in Section 3 and show that the test that maximizes the power for the alternative $H(\cdot/\Delta)$ maximizes the minimum power $\inf_F \beta(\phi; F; \Delta)$ over the class of F that are more IFRA than H . Since the rank tests that are optimal for the Weibull distribution

$$\hat{H}(t) = 1 - \exp(-at^b), \quad a, b > 0; \quad t \geq 0$$

coincides with the optimal tests for the exponential distribution (Lehmann (1953)), it follows that the tests in Sections 3 and 4 are also minimax for the class of F that are more IFRA than \hat{H} . Note that $-\log[1-\hat{H}(t)]/t = at^{b-1}$, thus \hat{H} is IFRA for $b \geq 1$ and DFRA for $0 < b \leq 1$.

7. COMBINATION OF INDEPENDENT TEST STATISTICS. GOODNESS OF FIT

Let T_1, \dots, T_n be independent and such that T_1 has the continuous distribution function $G_1(\cdot)$. Let $F_1(\cdot), \dots, F_n(\cdot)$ be n given continuous distribution functions, then the problem is to test $H_0: G_1(\cdot) = F_1(\cdot), i = 1, \dots, n$, against $H_1: F_1(\cdot) \geq G_1(\cdot), i = 1, \dots, n$, with strict inequality at least once. In the goodness of fit problem, it is usually assumed that $F_1 = \dots = F_n = F$ and $G_1 = \dots = G_n = G$. The goodness of fit problem considered is not the one in which the "shape" of the distribution is of interest, but where one is interested in testing whether a new procedure leads to stochastically larger variables T_1, \dots, T_n than a standard procedure in which the variables have known distributions F_1, \dots, F_n . For references to the "combination of independent test statistics" problem, see van Zwet and Oosterhoff (1967).

A test ϕ is called *monotone* if

$$(7.1) \quad \begin{aligned} & t'_1 \geq t_1, i = 1, \dots, n, \text{ implies} \\ & \phi(t_1, \dots, t_n) \leq \phi(t'_1, \dots, t'_n) \end{aligned}$$

and it is called *strongly distribution-free (SDF)* if its distribution depends on (F_1, G_1) only through $G_1 F_1^{-1}(\cdot)$, where $G_1 F_1^{-1}(t) = P(F_1(T_1) \leq t | G_1)$. Let $\beta(\phi; \tilde{F}; \tilde{\Delta})$ denote the power of the test ϕ for the IFRA scale alternative in which $G_1(\cdot) = F_1(\cdot/\Delta_1)$, and $F_1(\cdot)$ is an IFRA distribution. Let $K_1(x) = 1 - \exp(-x)$, $x \geq 0$, and let $\tilde{\Delta} \leq \tilde{1}$ ($\tilde{\Delta} \geq \tilde{1}$) mean $\Delta_1 \leq 1$ ($\Delta_1 \geq 1$), $i = 1, \dots, n$, then

Theorem 7.1

If ϕ is a monotone strongly distribution-free test, then

- (i) for each $\tilde{\Delta} \leq \tilde{1}$, $\beta(\phi; \tilde{F}; \tilde{\Delta}) \leq \beta(\phi; \tilde{K}; \tilde{\Delta})$, and
- (ii) for each $\tilde{\Delta} \geq \tilde{1}$, $\beta(\phi; \tilde{K}; \tilde{\Delta}) \leq \beta(\phi; \tilde{F}; \tilde{\Delta})$

Proof:

Set $T_1' = -\log[1-F_1(T_1)]$. Then the SDF property implies that $\beta(\phi; \tilde{F}; \tilde{\Delta})$ equals the power for testing H_0' : " T_1' has the exponential distribution K " vs H_1' : " T_1' has the distribution $G_1^*(\cdot) = F_1\{F_1^{-1}[K(\cdot)]/\Delta\}$ ". Using the transformation $T_1'' = -\Delta \log[1-F(T_1/\Delta)]$, the IFRA property, and the arguments of Section 2, one finds that $G_1^*(\cdot) \leq K(\cdot/\Delta_1)$ whenever $\Delta_1 \geq 1$, $i = 1, \dots, n$. Since monotone SDF tests have monotone power (Chapman (1958)), (ii), follows. The proof for $\Delta_1 \leq 1$, $i = 1, \dots, n$ is similar.

Fisher's test ϕ_F which rejects for large values of the statistic

$$(7.2) \quad - \sum_{i=1}^n \log[1-F_1(T_1)]$$

is strongly distribution-free and monotone. It is also uniformly most powerful for the Lehmann alternatives $G_1(t) = 1 - [1-F_1(t)]^{1/\theta}$, $\theta > 1$ (which includes the exponential and Weibull scale alternatives). It will next be shown that it is minimax for the scale model when $\Delta_1 = \dots = \Delta_n$.

Theorem 7.2

(i) Fisher's test ϕ_F is minimax for the IFRA scale alternative, i.e., if each F_1 ranges over the class of all continuous IFRA distributions, then whenever $\Delta_1 = \dots = \Delta_n > 1$,

$$(7.3) \quad \inf \beta(\phi; \tilde{F}; \tilde{\Delta}) \leq \inf \beta(\phi_F; \tilde{F}; \tilde{\Delta})$$

for all level α tests ϕ .

(ii) If ϕ' is another level α SDF test satisfying (7.3), then $\phi' = \phi_F$ a.e.

Proof:

Theorem 7.1 implies that

$$(7.4) \quad \inf \beta(\phi_F; \tilde{F}; \tilde{\Delta}) = \beta(\phi_F; \tilde{K}; \tilde{\Delta}).$$

Using the Neyman-Pearson Lemma, one finds that

$$(7.5) \quad \beta(\phi; \tilde{K}; \tilde{\Delta}) \leq \beta(\phi_F; \tilde{K}; \tilde{\Delta})$$

for all ϕ such that $\beta(\phi; K; 1) \leq \alpha$. Since

$$(7.6) \quad \inf \beta(\phi; \tilde{F}; \tilde{\Delta}) \leq \beta(\phi; \tilde{K}; \tilde{\Delta}),$$

the result (7.3) follows. The uniqueness part follows from the uniqueness part of the Neyman-Pearson Lemma and the structure (d) property of SDF statistics derived by Birnbaum and Rubin (1954) and Bell (1960). The structure (d) result implies that all SDF statistics are functions of T_1, \dots, T_n only through $F_1(T_1), \dots, F_n(T_n)$.

Remarks

(i) Note that the minimax property is uniform in $\Delta = \Delta_1 = \dots = \Delta_n$ and that it holds in the class of all tests satisfying $\beta(\phi; \tilde{K}; 1) \leq \alpha$ (not just the class of SDF level α tests).

(ii) If the Δ 's are not all equal, then the minimax statistic is

$$- \sum_{i=1}^n [(\Delta_i - 1)/\Delta_i] \log [1 - F_i(T_i)].$$

A sequential test is monotone if $t'_1 \geq t_1, i = 1, 2, \dots$, implies $\phi(t_1, t_2, \dots) \leq \phi(t'_1, t'_2, \dots)$. SDF tests are defined as before. Theorem 7.1 holds also for monotone SDF sequential tests. One such test is the test $\hat{\phi}_{\Delta_1}$ which at the n th stage rejects H_0 if $T_F > \log b$, accepts H_0 if $T_F < \log a$, and continuous sampling of $\log a \leq T_F < \log b$, where

$$(7.7) \quad T_F = -[(\Delta_1 - 1)/\Delta_1] \sum_{i=1}^n [1 - F_i(T_i)]$$

$$-n \log \Delta_1, \Delta_1 > 1, \text{ and } a < 1 < b.$$

This is the sequential probability ratio test for the Lehmann alternatives, moreover, it coincides with the sequential probability ratio test for exponential scale alternatives. Let $E[\cdot|\tilde{F},\Delta]$ denote the expected value when $F_1(\cdot), F_2(\cdot), \dots$ are the H_0 distributions of T_1, T_2, \dots and $F_1(\cdot/\Delta), F_2(\cdot/\Delta), \dots$ are the H_1 distributions. If N denotes the number of stages before termination, then the optimality property of sequential probability ratio tests and the above implies that

Theorem 7.3

If a and b are such that $E[\hat{\phi}_{\Delta_1}|\tilde{K},1] = \alpha$ and $E[\hat{\phi}_{\Delta_1}|\tilde{K},\Delta_1] = \beta$, then $\hat{\phi}_{\Delta_1}$ minimizes $E[N|\tilde{K},1]$ and $E[N|\tilde{K},\Delta_1]$ in the class of all tests ϕ satisfying $E[\phi|\tilde{K},1] \leq \alpha$, $E[\phi|\tilde{K},\Delta_1] \geq \beta$, $E[N|\tilde{K},1] < \infty$, and $E[N|\tilde{K},\Delta_1] < \infty$. Moreover, when $F_1(\cdot), F_2(\cdot), \dots$ are continuous IFRA distributions, then $E[\hat{\phi}_{\Delta_1}|\tilde{F},\Delta] \leq \alpha$ for each $\Delta \leq 1$, and $E[\hat{\phi}_{\Delta_1}|\tilde{F},\Delta] \geq \beta$ for each $\Delta \geq \Delta_1$.

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